# Solutions Manual

# Engineering Optimization

## Theory and Practice

### Fourth Edition

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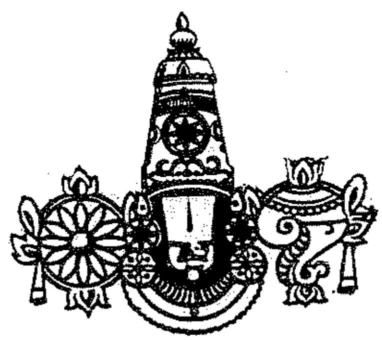

# Contents

# Chapter 1
# Introduction to Optimization

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**1.1** Let $x_A, x_B, x_C, x_D$ denote the quantity of fertilizer A, B, C, D produced per week (tons).

Let $P_i$ = profit per ton of fertilizer $i$ ($i$ = A, B, C, D)

$$P_i = SP_i - PC_i - (c_N x_{N_i} + c_{Ph} x_{Ph_i} + c_{Po} x_{Po_i} + c_{ch} x_{ch_i})$$

where $SP_i$ = selling price of fertilizer $i$ per ton

$PC_i$ = production cost of fertilizer $i$ per ton

$c_N$ = cost of nitrates per ton = \$ 1500

$c_{Ph}$ = cost of phosphates per ton = \$ 500

$c_{Po}$ = cost of potash per ton = \$ 1000

$c_{ch}$ = cost of inert chalk base = \$ 100

$x_{Ph_i}$ = tons of nitrate required for 1 ton of fertilizer $i$

$x_{Ph_i}$ = tons of phosphates required for 1 ton of fertilizer $i$

$x_{Po_i}$ = tons of potash required for 1 ton of fertilizer $i$

$x_{ch_i}$ = tons of inert chalk base required for 1 ton of ferlilizer $i$

$$P_A = 350 - 100 - (1500 * 0.05 + 500 * 0.1 + 1000 * 0.05 + 100 * 0.8) = -5$$

$$P_B = 550 - 150 - (1500 * 0.05 + 500 * 0.15 + 1000 * 0.1 + 100 * 0.7) = 80$$

$$P_C = 450 - 200 - (1500 * 0.1 + 500 * 0.2 + 1000 * 0.1 + 100 * 0.6) = -160$$

$$P_D = 700 - 250 - (1500 * 0.15 + 500 * 0.05 + 1000 * 0.15 + 100 * 0.65) = -15$$

Minimize

$$f = -\text{profit} = -P_A x_A - P_B x_B - P_C x_C - P_D x_D$$

$$\therefore f = 5x_A - 80x_B + 160x_C + 15x_D \qquad (1)$$

Availability constraints:

$$0.05x_A + 0.05x_B + 0.1x_C + 0.15x_D \le 1000 \quad \text{(nitrates)} \qquad (2)$$

$$0.1x_A + 0.15x_B + 0.2x_C + 0.05x_D \le 2000 \quad \text{(phosphates)} \qquad (3)$$

$$0.05x_A + 0.1x_B + 0.1x_C + 0.15x_D \le 1500 \quad \text{(potash)} \qquad (4)$$

Production requirements:

$$x_A \ge 5000, \quad x_B \ge 0, \quad x_C \ge 0, \quad x_D \ge 4000 \qquad (5)$$

$\therefore$ Problem is defined by Eqs. (1) to (5).

**1.2** $\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A/A_{ref} \\ x/h \end{Bmatrix}$

Objective functions:

$$f_1(\vec{X}) = 2\rho h\, x_2 \sqrt{1+x_1^2}\, A_{ref} \qquad \text{(weight)}$$

$$f_2(\vec{X}) = \frac{P h\, (1+x_1^2)^{1.5} \sqrt{1+x_1^4}}{2\sqrt{2}\, E\, x_1^2\, x_2\, A_{ref}} \qquad \text{(deflection)}$$

Behavior constraints:

$$\sigma_1(\vec{X}) \le \sigma_0$$

$$\sigma_2(\vec{X}) \le \sigma_0$$

with

$$\sigma_1 = \frac{P\,(1+x_1)\sqrt{1+x_1^2}}{2\sqrt{2}\, x_1 x_2\, A_{ref}}, \qquad \sigma_2 = \frac{P\,(x_1-1)\sqrt{1+x_1^2}}{2\sqrt{2}\, x_1 x_2\, A_{ref}}$$

Side constraints:

$$x_i^{min} \le x_i \le x_i^{max}, \quad i = 1, 2$$

Data: $E = 30 \times 10^6$ psi, $\rho = 0.283$ lb/in$^3$, $P = 10000$ lb,

$\sigma_0 = 20000$ psi, $A_{ref} = 1$ in$^2$, $h = 100$ in,

$x_1^{min} = x_2^{min} = 0.1$, $x_1^{max} = 2.0$, $x_2^{max} = 2.5$

(a) For given data,

$$f_1(\vec{X}) = 56.6\; x_2 \sqrt{1+x_1^2} \qquad (1)$$

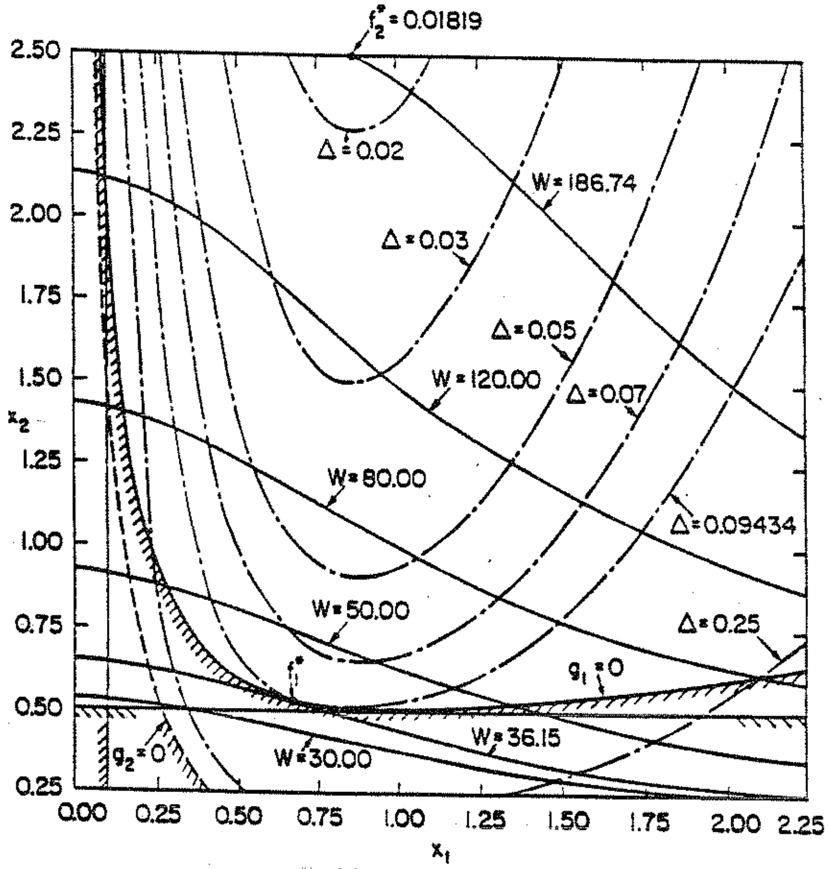
$$g_1(\vec{X}) \Rightarrow \frac{(1+x_1)\sqrt{1+x_1^2}}{x_1 x_2} \le 5.656854 \qquad (2)$$

$$g_2(\vec{X}) \Rightarrow \frac{(x_1-1)\sqrt{1+x_1^2}}{x_1 x_2} \le 5.656854 \qquad (3)$$

$$0.1 \le x_1 \le 2, \quad 0.1 \le x_2 \le 2.5 \qquad (4)$$

Graphical optimization shown in the figure.

Optimum solution: $\vec{X}^* = \begin{Bmatrix} 0.65 \\ 0.53521 \end{Bmatrix}$, $f_1^* = 36.12969$




$f_1 = W = \text{weight}$

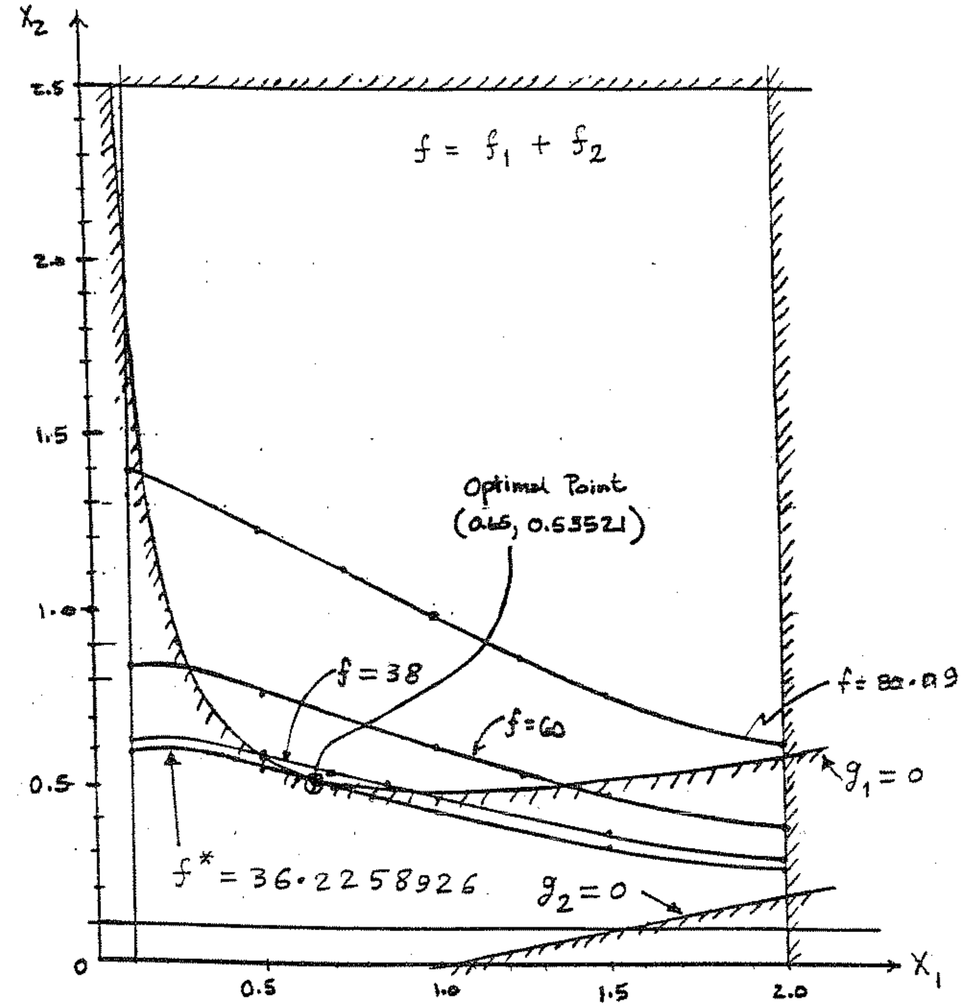
$f_2 = \Delta = \text{deflection}$

(b)
$$f_2(\vec{x}) = \frac{0.011785\,11\,(1+x_1^2)^{1.5}\sqrt{1+x_1^4}}{x_1^2\,x_2} \qquad (5)$$

constraints are given by Eqs. (2), (3) and (4).

Optimum solution: $\vec{x}^* = \begin{Bmatrix} 0.9 \\ 2.5 \end{Bmatrix}$, $f_2^* = 0.01823$

(Graphical solution is indicated in the above figure).




(c) $f_1(\vec{x}) + f_2(\vec{x})$

$$= 56.6\ x_2 \sqrt{1 + x_1^2} + \frac{0.01178511\,(1 + x_1^2)^{1.5} \sqrt{1 + x_1^4}}{x_1^2\, x_2} \qquad (6)$$

Constraints are given by Eqs. (2), (3) and (4).

Optimum solution:

$$\vec{X}^* = \begin{Bmatrix} 0.65 \\ 0.53521 \end{Bmatrix}, \quad f_3^* = 36.2258926$$

(Graphical solution is indicated in the above figure).

**1.3** Jobs: $i = 1, 2, \ldots, 10$

Stations: $j = 1, 2, \ldots, n$

$t_i$ = time required for job $i$ = $t_{fi} - t_{si}$ , $i = 1, 2, \ldots, 10$

where $t_{fi}$ = time at which job $i$ is completed,

$t_{si}$ = time at which job $i$ is started

Let $t_E$ = time at which all jobs are completed.

For station $j$, idle time = $I_i = t_E - \sum_{i=1}^{10} x_{ij} t_i$

where $x_{ij} = \begin{cases} 1 & \text{if job } i \text{ is assigned to station } j \\ 0 & \text{otherwise} \end{cases}$

$$\text{Total idle time} = f = \sum_{j=1}^{n} I_j = \sum_{j=1}^{n} \sum_{i=1}^{10} \left( t_E - x_{ij} t_i \right) \qquad (1)$$

At station $j$,

$$x_{ij} t_{si} = \sum_{k=1}^{i-1} x_{kj} t_k \; ; \; j = 1, 2, \ldots, n \qquad (2)$$

For job $i$,

$$t_{fi} = t_i + t_{si} \; ; \; i = 1, 2, \ldots, 10 \qquad (3)$$

Find $n, x_{11}, x_{12}, \ldots, x_{10,n}$

to minimize $f$ given by Eq. (1)

subject to

$t_{s5} \geq t_{f1}$ , $f_{s5} \geq t_{f3}$ ,

$t_{s6} \geq t_{f2}$ , $t_{s6} \geq t_{f3}$ , $t_{s6} \geq t_{f4}$ ,

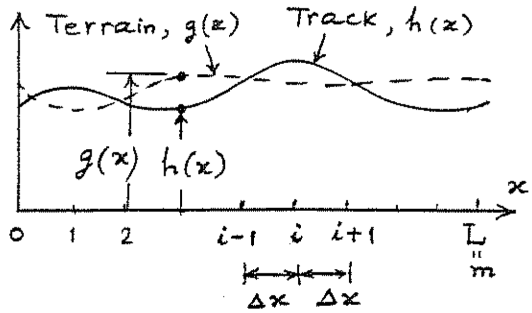
$t_{s7} \geq t_{f5}$ , $t_{s7} \geq t_{f6}$ ,

$t_{s8} \geq t_{f6}$ , $t_{s9} \geq t_{f7}$ , $t_{s9} \geq t_{f8}$ , $t_{s10} \geq t_{f9}$

and Eqs. (2) and (3)

with $t_1 = 4$, $t_2 = 8$, $t_3 = 7$, $t_4 = 6$, $t_5 = 3$,
$t_6 = 5$, $t_7 = 1$, $t_8 = 9$, $t_9 = 2$, $t_{10} = 8$

**1.4** Divide the track length into m intervals of equal length as shown in the figure:




Design variables:

$$\{h_0, h_1, h_2, \ldots, h_m\} \qquad (1)$$

with $h_i = h\,(x = x_i)$

and $x_i = i.(\Delta x)$ ; $\Delta x = (L/m)$.

objective function: (to be minimized)

$$f = c \int_0^L |g(x) - h(x)|\, dx$$

$$\approx c \sum_{i=1}^{m} |g(x_i) - h_i| \cdot \Delta x \qquad (2)$$

where c is a proportionality constant.

Constraints:

slope:

$$\left|\frac{h_{i+1} - h_i}{\Delta x}\right| \le r_1 \quad ; \; i = 0, 1, \ldots, m-1 \qquad (3)$$

Rate of change of slope:

$$\left|\frac{\dfrac{h_{i+2} - h_{i+1}}{\Delta x} - \dfrac{h_{i+1} - h_i}{\Delta x}}{\Delta x}\right| = \left|\frac{h_{i+2} - 2h_{i+1} + h_i}{(\Delta x)^2}\right| \le r_2 \;;$$

$$i = 0, 1, \ldots, m-2 \qquad (4)$$

second derivative of slope:

$$\left|\frac{\dfrac{h_{i+3} - 2h_{i+2} + h_{i+1}}{(\Delta x)^2} - \dfrac{h_{i+2} - 2h_{i+1} + h_i}{(\Delta x)^2}}{\Delta x}\right|$$

$$= \left|\frac{h_{i+3} - 3h_{i+2} + 3h_{i+1} - h_i}{(\Delta x)^3}\right| \le r_3 \;;\; i = 0, 1, \ldots, m-3 \qquad (5)$$

End conditions:

$$h_0 = a \;, \quad h_m = b \qquad (6)$$

Problem defined by Eqs. (1) to (6) is an O.C. problem. (It has serial structure).

**1.5** Let $x_i$ = number of units produced in week $i$ $(i=1,2)$

Assume: (1) Every thing produced is sold by end of second week.

(2) More units can be produced in 1st week, if economical.

(3) Total units in the two weeks is 600.

Minimize $f = x_1(4x_1^2) + x_2(4x_2^2) + 10(x_1 - 200)$

subject to

$$200 \le x_1 \le 600$$
$$0 \le x_2 \le 600$$
$$x_1 + x_2 = 600$$

Graphical solution:

Contours of $f$ are drawn as follows:

$$f = 4x_1^3 + 4x_2^3 + 10x_1 - 2000 = c = \text{constant}$$

$c = 6 \times 10^8$ :

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_2$ | 100 | 200 | 300 | 400 | 500 | 600 |
| $x_1$ | 530.1 | 521.7 | 497 | 441 | 292 | — |

$c = 4 \times 10^8$ :

| | | | | | |
|---|---|---|---|---|---|
| $x_2$ | 100 | 200 | 300 | 400 | 500 |
| $x_1$ | 463 | 451 | 417 | 330 | −292 |

$c = 3 \times 10^8$ :

| | | | | | |
|---|---|---|---|---|---|
| $x_2$ | 100 | 200 | 300 | 400 | 500 |
| $x_1$ | 420 | 406 | 363.4 | 222.4 | −368 |

$c = 2.75 \times 10^8$ :

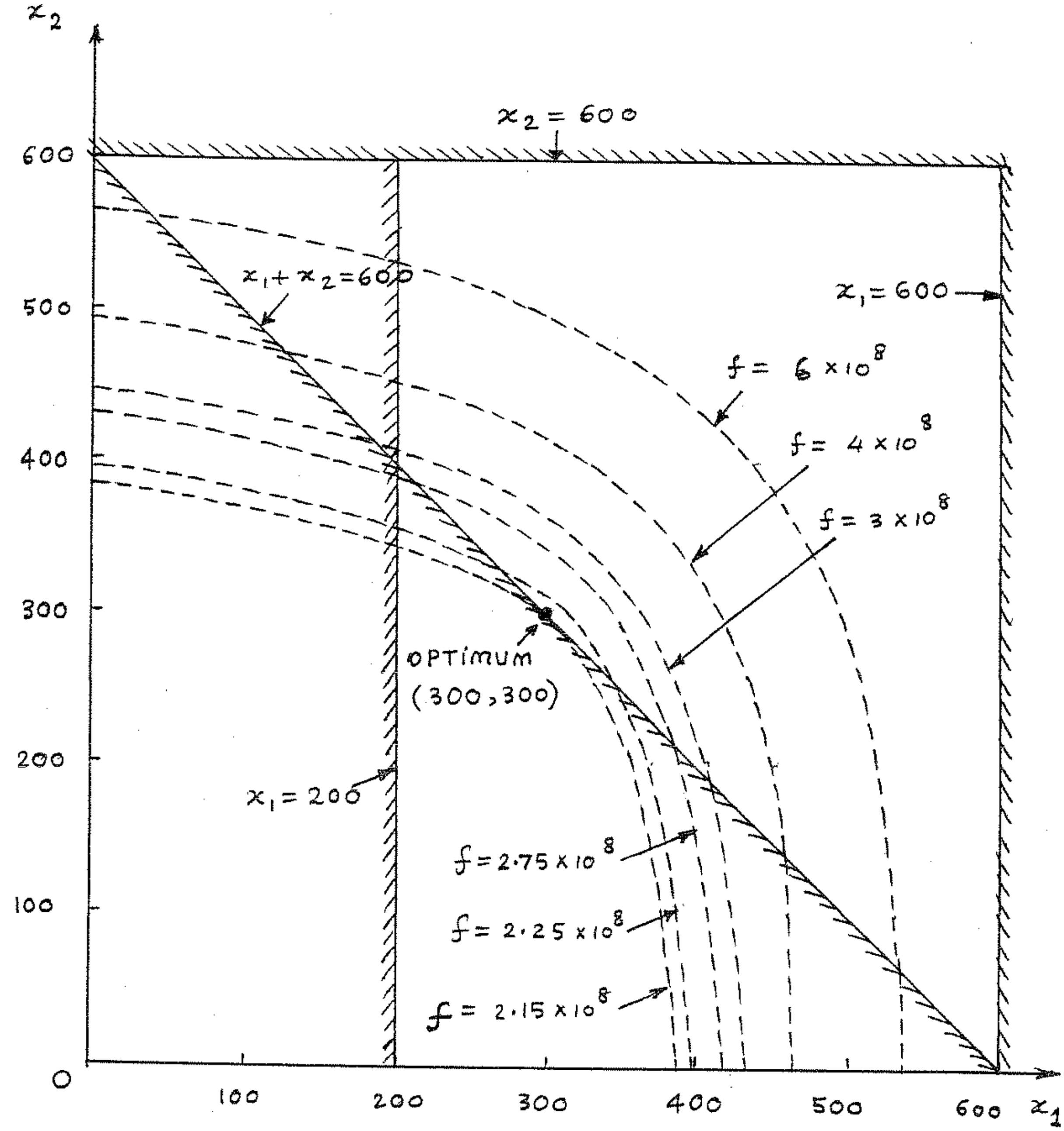
| | | | | | |
|---|---|---|---|---|---|
| $x_2$ | 100 | 200 | 300 | 400 | 500 |
| $x_1$ | 407 | 393 | 346 | 168 | −383 |

$c = 2.25 \times 10^8$ :

| | | | | |
|---|---|---|---|---|
| $x_2$ | 100 | 200 | 300 | 400 |
| $x_1$ | 380 | 364 | 308 | −197 |

$c = 2.15 \times 10^8$ :

| | | | | |
|---|---|---|---|---|
| $x_2$ | 100 | 200 | 300 | 400 |
| $x_1$ | 375 | 358 | 299 | −217 |



$x_2$
$x_2 = 600$
600
500
400
300
200
100
0
100
200
300
400
500
600
$x_1$
$x_1 + x_2 = 600$
$x_1 = 600$
$f = 6 \times 10^8$
$f = 4 \times 10^8$
$f = 3 \times 10^8$
OPTIMUM
(300, 300)
$x_1 = 200$
$f = 2.75 \times 10^8$
$f = 2.25 \times 10^8$
$f = 2.15 \times 10^8$

**1.6** $\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} r \\ \ell \end{Bmatrix}$

$$x = x_1 \cos\theta + x_2 \sqrt{1 - \left(\frac{x_1 \sin\theta}{x_2}\right)^2} \tag{1}$$

$$\dot{x} = \frac{dx}{dt} = -x_1 \omega \left\{ \sin\theta + \frac{x_1 \sin 2\theta}{2 x_2 \sqrt{1 - \left(\frac{x_1 \sin\theta}{x_2}\right)^2}} \right\} \tag{2}$$

where $\omega = \frac{d\theta}{dt} = \dot{\theta}$

$$\ddot{x} \simeq -x_1 \omega^2 \left(\cos\theta + \frac{x_1}{x_2} \cos 2\theta\right) \tag{3}$$

To plot the constraint on $x$, we rearrange Eq. (1) as:

$$(x - x_1 \cos\theta)^2 = x_2^2 \left(1 - \frac{x_1^2}{x_2^2} \sin^2\theta\right) = x_2^2 - x_1^2 \sin^2\theta$$

or
$$x_2 = \left\{ (x_1 \sin\theta)^2 + (x - x_1 \cos\theta)^2 \right\}^{\frac{1}{2}} \tag{4}$$

To plot the contours of $\dot{x}$, Eq. (2) is rewritten as:

$$(\dot{x} + x_1 \omega \sin\theta)^2 = - \frac{x_1^2 \omega \sin 2\theta}{2 x_2 \sqrt{1 - \frac{x_1^2 \sin^2\theta}{x_2^2}}}$$

or
$$(\dot{x} + x_1 . \omega . \sin\theta)^2 = \frac{(x_1^2 \omega \sin 2\theta)^2}{4 (x_2^2 - x_1^2 \sin^2\theta)}$$

or
$$x_2 = \sqrt{\frac{1}{4} \left(\frac{x_1^2 \omega \sin 2\theta}{\dot{x} + x_1 \omega \sin\theta}\right)^2 + (x_1 \sin\theta)^2} \tag{5}$$

To plot the contours of $\ddot{x}$, Eq. (3) is rewritten as:

$$x_2 (\ddot{x} + x_1 \omega^2 \cos\theta) = - x_1^2 \omega^2 \cos 2\theta$$

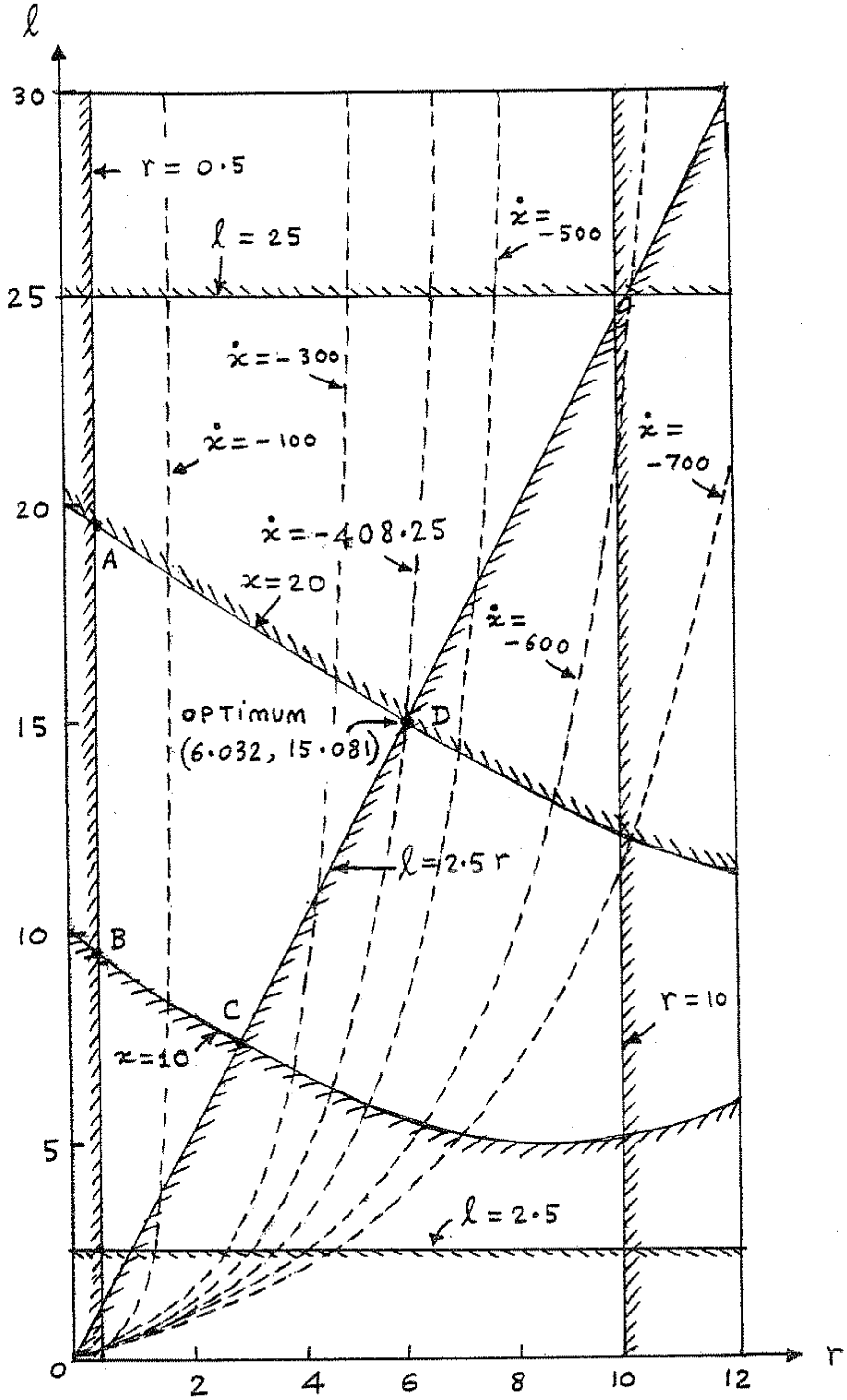
or
$$x_2 = \frac{- x_1^2 \omega^2 \cos 2\theta}{\ddot{x} + x_1 \omega^2 \cos\theta} \tag{6}$$

Data: $\theta = 30^\circ$, $\omega = 100$ rad/s

Constraints: $\ell \geq 2.5\, r \Rightarrow \frac{x_2}{x_1} \geq 2.5$

$0.5 \leq x_1 \leq 10$, $2.5 \leq x_2 \leq 25$

$10 \leq x \leq 20$

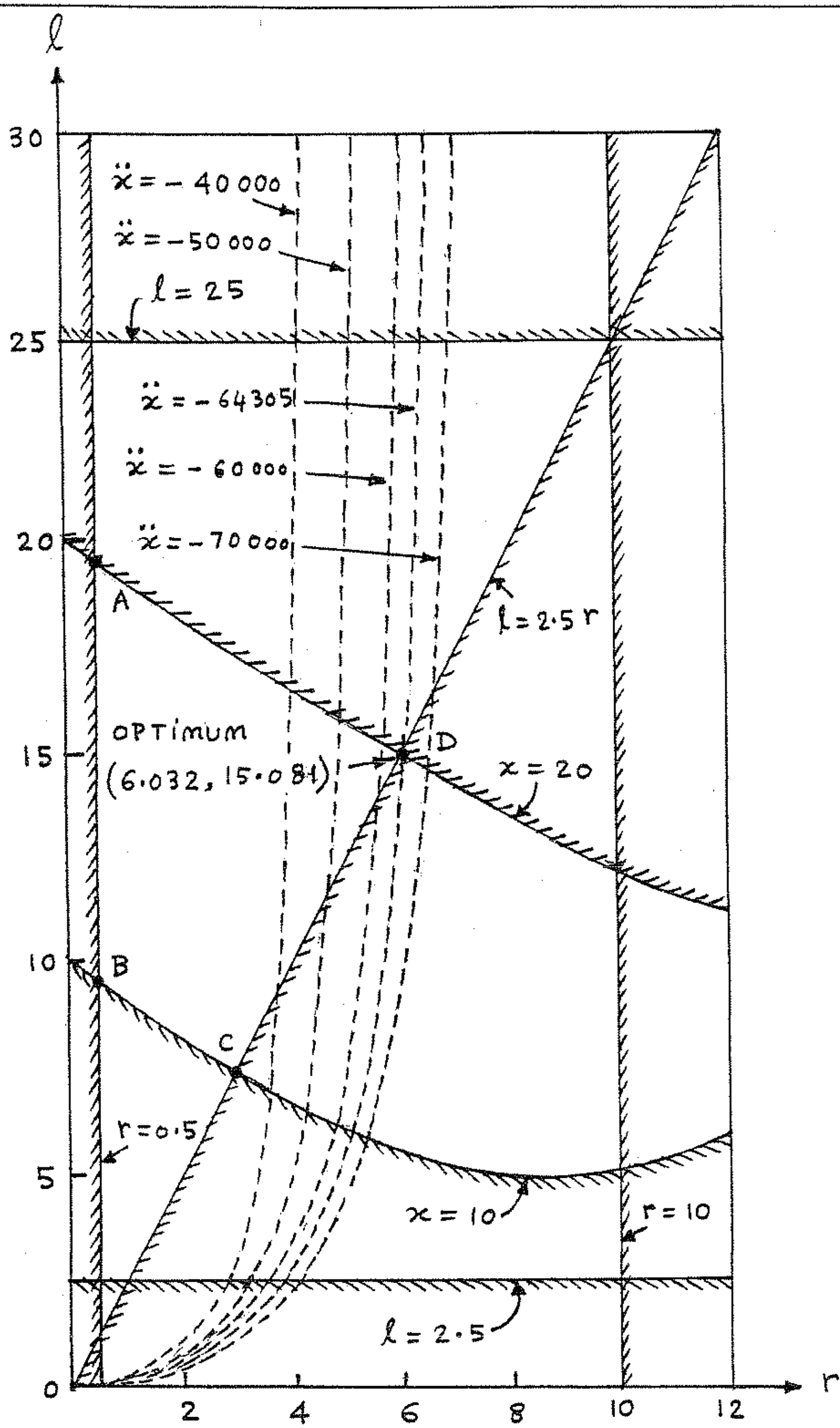



Feasible space = ABCDA

Optimum solution :

$r^* = 6.032$, $l^* = 15.081$, $\dot{x}_{opt} = -408.25$

**1.7**




Feasible space = ABCDA

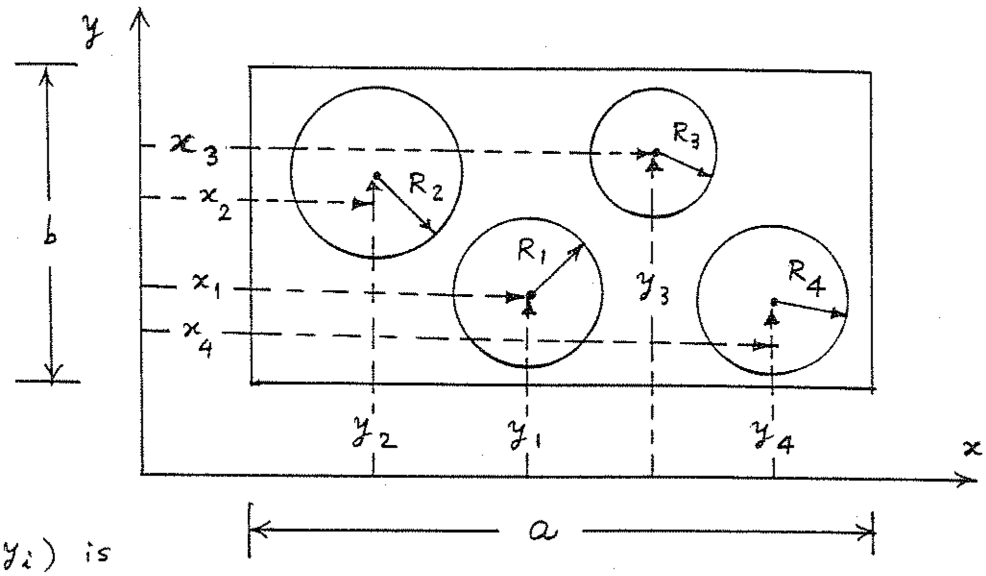
optimum solution: $r^* = 6.032$, $\ell^* = 15.081$

$\ddot{x}_{opt} = -64\,305$

(Formulation: given in solution of problem 1.7)

**1.8**

$$\vec{X} = \begin{Bmatrix} a \\ b \\ x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$




where $(x_i, y_i)$ is the center of disk $i$.

Minimize $f = ab - \pi(R_1^2 + R_2^2 + R_3^2 + R_4^2)$

subject to the constraints

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \geq R_1 + R_2$$

$$\sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} \geq R_1 + R_3$$

$$\sqrt{(x_1 - x_4)^2 + (y_1 - y_4)^2} \geq R_1 + R_4$$

$$\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} \geq R_2 + R_3$$

$$\sqrt{(x_2 - x_4)^2 + (y_2 - y_4)^2} \geq R_2 + R_4$$

$$\sqrt{(x_3 - x_4)^2 + (y_3 - y_4)^2} \geq R_3 + R_4$$

$$a \geq x_i + R_i \ ; \quad i = 1, 2, 3, 4$$

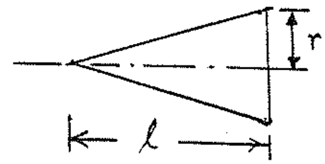
$$b \geq y_i + R_i \ ; \quad i = 1, 2, 3, 4$$

$$x_i \geq R_i \ ; \quad i = 1, 2, 3, 4$$

$$y_i \geq R_i \ ; \quad i = 1, 2, 3, 4$$

**1.9** $\vec{X} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$

(a) Volume of a cone $= \frac{1}{3}\pi r^2 \ell$




Volume of cone clutch:

$$f(\vec{X}) = \frac{1}{3}\pi\left(R^2\ell_1 - R_2^2\ell_2\right)$$

$$= \frac{1}{3}\frac{\pi}{\tan\alpha}\left(R_1^3 - R_2^3\right) \qquad (1)$$

$$T = \frac{2\pi f p}{3\sin\alpha}\left(R_1^3 - R_2^3\right) \qquad (2)$$

Using $p = \frac{F}{A}$, $A = \pi\left(R_1^2 - R_2^2\right)$, Eq. (2) becomes

$$T = \frac{2\pi\ (0.5)\quad 30}{3\ (0.5)\quad \pi\left(R_1^2 - R_2^2\right)}\left(R_1^3 - R_2^3\right) = 20\left(\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right) \qquad (3)$$

Constraints:

$$g_1(\vec{X}) = \frac{R_1^3 - R_2^3}{R_1^2 - R_2^2} - 5 \geq 0 \qquad (4)$$

$$g_2(\vec{X}) = R_1 - 2R_2 \geq 0 \qquad (5)$$

$$g_3(\vec{X}) = R_1 \geq 0 \qquad (6)$$

$$g_4(\vec{X}) = R_1 - 15 \leq 0 \qquad (7)$$

$$g_5(\vec{X}) = R_2 \geq 0 \qquad (8)$$

$$g_6(\vec{X}) = R_2 - 10 \leq 0 \qquad (9)$$

Plotting $g_1 = 0$:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $R_1$ | 0.0 | 1.0 | 2.0 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
| $R_2$ | 5.0 | 4.83 | 4.37 | 3.65 | 3.16 | 2.56 | 1.77 | 0.0 |

Contours of $f$ (Eq. 1):

For $f = 50.0$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $R_1$ | 3.02 | 3.5 | 4.0 | 5.0 | 6.0 | 7.0 | 10.0 |
| $R_2$ | 0.0 | 2.48 | 3.32 | 4.6 | 5.73 | 6.8 | 9.9 |

for $f = 100.0$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $R_1$ | 3.81 | 4.0 | 4.5 | 5.0 | 6.0 | 7.0 | 10.0 |
| $R_2$ | 0.0 | 2.07 | 3.3 | 4.1 | 5.4 | 6.6 | 9.8 |

for $f = 141.9$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $R_1$ | 4.27 | 4.5 | 5.0 | 5.5 | 6.0 | 8.0 | 10.0 |
| $R_2$ | 0.0 | 2.34 | 3.6 | 4.45 | 5.16 | 7.6 | 9.73 |

for $f = 0.0$:

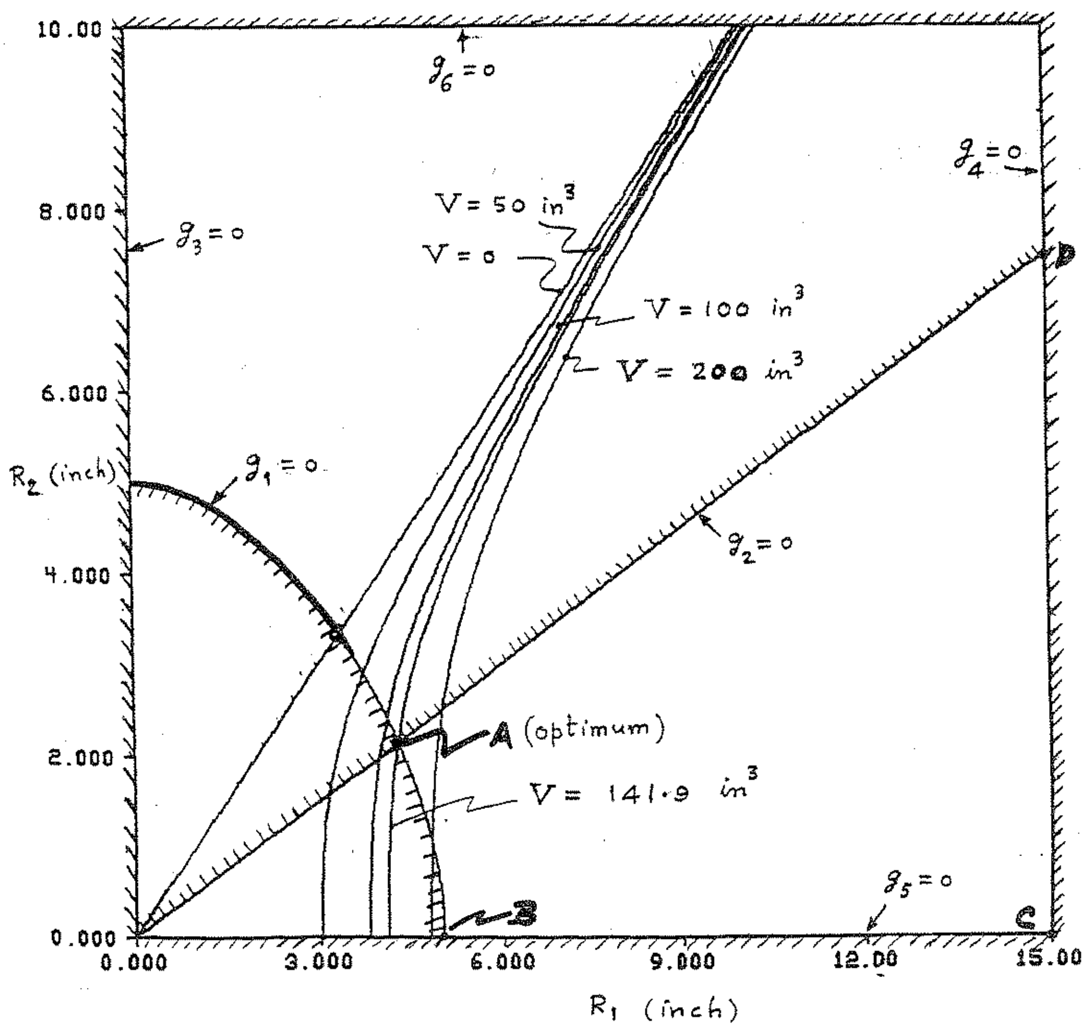
| | | | | | |
|---|---|---|---|---|---|
| $R_1$ | 1.0 | 2.0 | 3.0 | 5.0 | 10.0 |
| $R_2$ | 1.0 | 2.0 | 3.0 | 5.0 | 10.0 |

Graphical solution : indicated in the figure.

Optimum point: $\vec{X}^* = \begin{Bmatrix} R_1^* \\ R_2^* \end{Bmatrix} = \begin{Bmatrix} 4.472 \\ 2.236 \end{Bmatrix}$, $f^* = 141.9 \text{ in}^3$

(Feasible space: ABCDA)

$\Rightarrow$ Point A



(b) Constraints = same as in part (a) except that $g_2$ is changed to $R_1 - 2R_2 \le 0$.

Objective function = same as in part (a).

Feasible space = ADJKLA

optimum point = L

$$\vec{X}^* = \begin{Bmatrix} R_1^* \\ R_2^* \end{Bmatrix} = \begin{Bmatrix} 3.536 \\ 3.536 \end{Bmatrix}, \quad f^* = 0 \text{ in}^3$$

Details of graphical solution : shown in the figure.