Chapter 2

2.1 **a.**   Spring constant, k : The change in the force per unit length change of the spring.

b.   Coefficient of subgrade reaction, k':

Spring constant divided by the foundation contact area, $k' = \dfrac{k}{A}$

c.   Undamped natural circular frequency: $\omega_n = \sqrt{\dfrac{k}{m}}$ rad/s

where $m = \text{mass} = \dfrac{W}{g}$

d.   Undamped natural frequency: $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$ (in Hz)

Note: Circular frequency defines the rate of oscillation in term of radians per unit time; 2π radians being equal to one complete cycle of rotation.

e.   Period, T: The time required for the motion to begin repeating itself.

f.   Resonance: Resonance occurs when $\dfrac{\omega_n}{\omega} = 1$

g.   Critical damping coefficient: $c_c = 2\sqrt{km}$

where k = spring constant; $m = \text{mass} = \dfrac{W}{g}$

h.   Damping ratio: $D = \dfrac{c}{c_c} = \dfrac{c}{2\sqrt{km}}$

where c = viscous damping coefficient; c_c = critical damping coefficient

i.   Damped natural frequency:

$\omega_d = \omega_n\sqrt{1 - D^2}$

$f_d = \sqrt{1 - D^2}\, f_n$

2.2   Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 100,000$ kN/m

Mass of the machine + foundation, $m = \dfrac{W}{g} = \dfrac{400}{9.81} = 40.77 \ \dfrac{\text{kN}}{\text{m/s}^2}$

Natural frequency of undamped free vibration is [Eq. (2.19)]

$$f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{100,000}{40.77}} = \textbf{7.88 Hz}$$

From Eq. (2.18), $T = \dfrac{1}{f_n} = \dfrac{1}{7.88} = \textbf{0.127 s}$

2.3   Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 100,000$ kN/m

Static deflection of foundation is [Eq. (2.2)]

$$z_s = \dfrac{W}{k} = \dfrac{400}{100,000} = 4 \times 10^{-3}\, \text{m} = \textbf{4 mm}$$

2.4   External force to which the foundation is subjected, $Q = 35.6 \sin \omega t$ kN
$f = 13.33$ Hz
Weight of the machine + foundation, $W = 178$ kN
Spring constant, $k = 70,000$ kN/m

For this foundation, let time $t = 0$, $z = z_0 = 0$, $\dot{z} = v_0 = 0$

a.   Mass of the machine + foundation, $m = \dfrac{W}{g} = \dfrac{178}{9.81} = 18.145 \ \dfrac{\text{kN}}{\text{m/s}^2}$

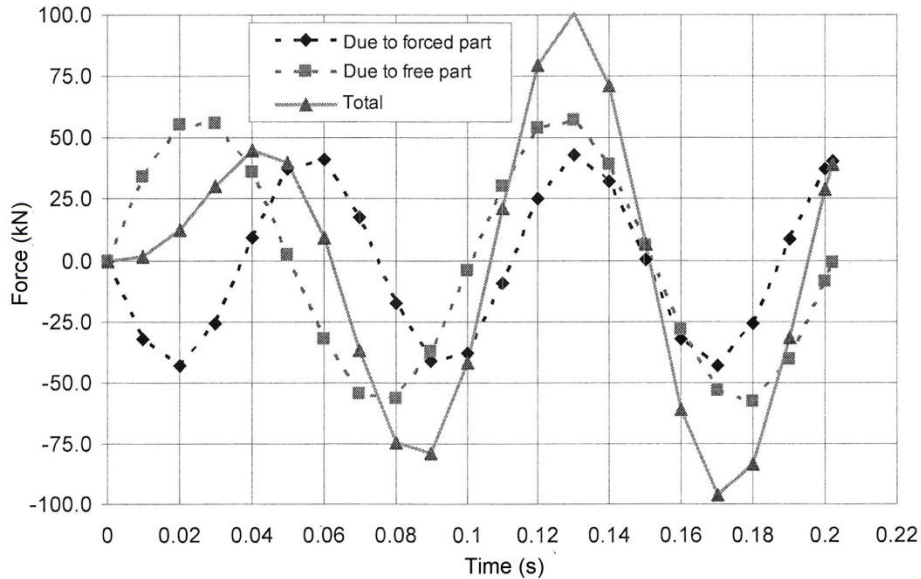$$\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{70,000}{18.145}} = 62.11 \ \text{rad/s}$$

$$T = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{62.11} = \textbf{0.101 s}$$

b.   The frequency of loading, $f = 13.33$ Hz

$$\omega = 2\pi f = 2\pi(13.33) = 83.75 \ \text{rad/s}$$

Force due to forced part, $F_1 = k\left(\dfrac{Q_0/k}{1-\omega^2/\omega_n^2}\right)\sin\omega t$

$$= (70{,}000)\left(\dfrac{35.6/70{,}000}{1-83.75^2/62.11^2}\right)\sin(83.75t)$$

$$= \mathbf{43.51\sin(83.75}\textbf{\textit{t}}\mathbf{)\ kN}$$

See the plot below for F_1 vs. t

c.   Force due to free part, $F_2 = k\left(\dfrac{Q_0/k}{1-\omega^2/\omega_n^2}\right)\left(-\dfrac{\omega}{\omega_n}\sin\omega_n t\right)$

$$= 70{,}000\left(\dfrac{35.6/70{,}000}{1-83.75^2/62.11^2}\right)\left(-\dfrac{83.75}{62.11}\sin(62.11t)\right)$$

$$= \mathbf{58.67\sin(62.11}\textbf{\textit{t}}\mathbf{)\ kN}$$

See the plot above in Part **b** for F_2 vs. t.

d.   Total dynamic force on the subgrade:

$$F = F_1 + F_2 = \mathbf{-43.51\sin(83.75}\textbf{\textit{t}}\mathbf{)+58.67(62.11}\textbf{\textit{t}}\mathbf{)\ kN}$$

The plot of variation of the dynamic force on the subgrade of the foundation due to (a) forced part, (b) free part, and (c) total of the response for time $t = 0$ to $t = 2T$ is shown in the figure above (Part **b**).

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2.5 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} \quad \text{where } k_{eq} = \text{equivalent stiffness of the spring system}$$

For springs attached in series, the equivalent stiffness is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}, \text{ or } \frac{1}{k_{eq}} = \frac{k_1 k_2}{k_1 + k_2}$$

The natural frequency of the given undamped free vibration spring mass system is

$$\boldsymbol{f_n = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{k_1 + k_2} \times \frac{1}{m}}}$$

2.6 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} \quad \text{where } k_{eq} = \text{equivalent stiffness of the spring system}$$

For springs attached in parallel, the equivalent stiffness is given by

$$k_{eq} = k_1 + k_2$$

The natural frequency of the given undamped free vibration spring mass system is

$$\boldsymbol{f_n = \frac{1}{2\pi}\sqrt{\frac{(k_1 + k_2)}{m}}}$$

2.7 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} \quad \text{where } k_{eq} = \text{equivalent stiffness of the spring system}$$
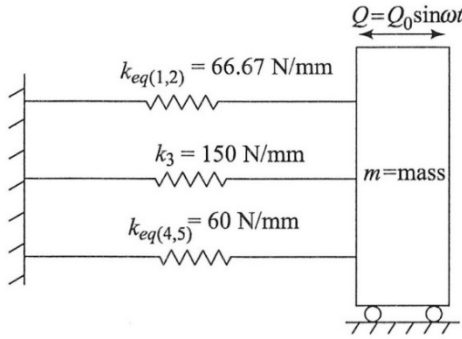
In the given spring-mass system, springs with stiffness k_1 and k_2 are in series.
Hence, their equivalent stiffness is

$$k_{eq(1,2)} = \frac{k_1 k_2}{k_1 + k_2} = \frac{100 \times 200}{100 + 200} = \frac{20,000}{300} = 66.67 \text{ N/mm}$$

Similarly, springs with stiffness k_4 and k_5 are in series. Hence, their equivalent stiffness is

$$k_{eq(4,5)} = \frac{k_4 k_5}{k_4 + k_5} = \frac{100 \times 150}{100 + 150} = 60 \text{ N/mm}$$

Now, the given spring system can be reduced to three springs in series.

The resulting system will be three springs in parallel. Their equivalent stiffness is given by

$$k_{eq} = k_{eq(1,2)} + k_3 + k_{eq(4,5)} = 66.67 + 150 + 60 = 276.67 \text{ N/mm} = 276.67 \text{ kN/m}$$

The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi} \sqrt{\frac{267.67 \times 1000}{100}} = \textbf{8.37 Hz}$$

Time period $T = (1/f_n) = (1/8.37) = \textbf{0.119 s}$

2.8   Sinusoidal–varying force, $Q = 50 \sin \omega t$ N; $Q_0 = 50$ N; $\omega = 47$ rad/s

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{276.67 \times 1000}{100}} = 52.6 \text{ rad/s}$$

Amplitude of vibration = static deflection z_s × magnification M

$$z_s = \frac{Q_0}{k_{eq}} = \frac{50}{276.67} = 0.1807 \text{ mm}$$

From Eq. (2.34),

$$M = \frac{1}{1 - (\omega/\omega_n)^2} = \frac{1}{1 - (47/52.6)^2} = 4.96$$

Amplitude of vibration = $0.1807 \times 4.96 = \textbf{0.896 mm}$

2.9   Weight of the body, $W = 135$ N
Mass of the body, $m = W/g = 135/9.81 = 13.76$ kg
Spring constant, $k = 2600$ N/m
Dashpot resistance, $c = 0.7/(60/1000) = 11.67$ N-s/m

a. Damped natural frequency [Eq. (2.67)]

$$f_d = \sqrt{1 - D^2}\, f_n$$

$$D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{11.67}{2\sqrt{2600 \times 13.76}} = 0.031$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2600}{13.76}} = 2.19\ \text{Hz}$$

$$f_d = \sqrt{1 - 0.031^2} \times (2.19) = \textbf{2.18 Hz}$$

b. Damping ratio [Eq. (2.47b)],

$$D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{11.67}{2\sqrt{2600 \times 13.76}} = \textbf{0.031}$$

c. Ratio of successive amplitudes of the body is given by [Eq. (2.70)],

$$\frac{Z_n}{Z_{n+1}} = e^{\delta}$$

where $\delta = \ln\left(\dfrac{Z_n}{Z_{n+1}}\right) = \dfrac{2\pi D}{\sqrt{1-D^2}} = \dfrac{2\pi \times 0.031}{\sqrt{1-0.031^2}} = 0.195$

$$\frac{Z_n}{Z_{n+1}} = e^{\delta} = e^{0.195} = \textbf{1.215}$$

d. At time $t = 0$ s, amplitude $Z_0 = 25$ mm.
After n cycles of disturbance

$$\frac{1}{n}\ln\frac{Z_0}{Z_n} = \frac{2\pi D}{\sqrt{1-D^2}} ; \quad \ln\frac{Z_0}{Z_n} = \frac{2\pi n D}{\sqrt{1-D^2}}$$

With $n = 5$,

$$\ln\frac{Z_0}{Z_5} = \frac{2\pi \times 5 \times D}{\sqrt{1-D^2}} = \frac{2\pi \times 5 \times 0.031}{\sqrt{1-0.031^2}} = 0.974$$

$$\frac{Z_0}{Z_5} = e^{0.974} = 2.649; \quad Z_5 = \frac{25}{2.649} = 9.44\ \text{mm}$$

After 5 cycles of disturbance, the amplitude of vibration = **9.44 mm**

2.10 $Q_0 = 6.7$ kN

$\omega = 3100$ rad/min $= 51.67$ rad/s

Weight of machine + foundation, $W = 290$ kN

Spring constant, $k = 875$ MN/m $= 875,000$ kN/m

Natural angular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{875,000 \times 10^3}{290 \times 10^3 / 9.81}} = 172.04$ rad/s

From Eq. (2.43), $F_{\text{dynam}} = \dfrac{Q_0}{1 - (\omega / \omega_n)} = \dfrac{6.7}{1 - (51.67 / 172.04)} = 9.58$ kN

Maximum force on the subgrade $= 290 + 9.58 = \textbf{299.58 kN}$

Minimum force on the subgrade $= 290 - 9.58 = \textbf{280.42 kN}$

2.11 $Q_0 = 200$ kN

$\omega = 6000$ rad/min $= 100$ rad/s

Weight of machine + foundation, $W = 400$ kN
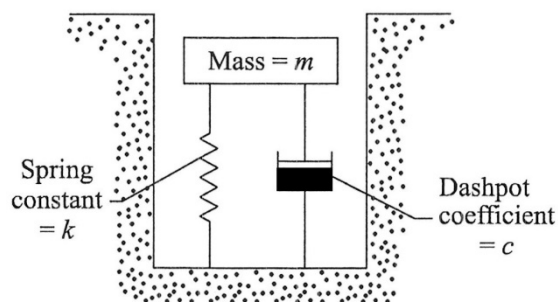
Spring constant, $k = 120,000$ kN/m

Natural angular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{120,000 \times 10^3}{400 \times 10^3 / 9.81}} = 54.25$ rad/s

Dynamic force, $F_{\text{dynam}} = \dfrac{Q_0}{1 - \omega / \omega_n} = \dfrac{200}{1 - (100 / 54.25)} = 237.16$ kN

Maximum force on the subgrade $= 400 + 237.16 = \textbf{637.16 kN}$

Minimum force on the subgrade $= 400 - 237.16 = \textbf{162.84 kN}$

2.12   Weight of the body, $W = 800$ kN

Spring constant, $k = 200,000$ kN/m

Dashpot coefficient, $c = 2340$ kN-s/m

a. $c_c = 2\sqrt{km} = 2\sqrt{200{,}000 \times 800/9.81} = \textbf{8077.1 kN-s/m}$

b. Damping ratio, $D = \dfrac{c}{c_c} = \dfrac{2340}{8077.1} = \textbf{0.29}$

c. $\delta = \dfrac{2\pi D}{\sqrt{1-D^2}} = \dfrac{2\pi \times 0.29}{\sqrt{1-0.29^2}} = \textbf{1.9}$

d. $f_d = \sqrt{1-D^2}\, f_n$; $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{200{,}000 \times 9.81}{800}} = 7.88$ Hz

$f_d = \sqrt{1-0.29^2} \times 7.88 = \textbf{7.54 Hz}$

2.13 Weight of the body, $W = 800$ kN
Spring constant, $k = 200{,}000$ kN/m
Dashpot coefficient, $c = 2340$ kN-s/m
$Q_0 = 25$ kN
Operating frequency, $\omega = 100$ rad/s

a. Natural circular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{200{,}000 \times 9.81}{800}} = 49.52$ rad/s

From Problem 2.12, damping ratio, $D = 0.29$

From Eq. (2.28), the amplitude of vertical vibration of the foundation is

$$Z = \dfrac{(Q_0/k)}{\sqrt{[1-(\omega^2/\omega_n^2)]^2 + 4D^2(\omega^2/\omega_n^2)}}$$

$$= \dfrac{(25/200{,}000)}{\sqrt{[1-(100^2/49.52^2)]^2 + 4 \times 0.29^2(100^2/49.52^2)}}$$

$$= \textbf{3.795} \times \textbf{10}^{-5}\,\textbf{m} = \textbf{3.795} \times \textbf{10}^{-2}\,\textbf{mm}$$

b. From Eq. (2.90), the maximum dynamic force transmitted to the subgrade is

$$Z\sqrt{k^2 + (c\omega)^2} = (3.795 \times 10^{-5})\sqrt{200{,}000^2 + (2340 \times 100)^2} = \textbf{11.68 kN}$$